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# Existence of multiple positive periodic solutions for functional differential equations

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**Abstract**

In our paper, by employing Krasnoselskii fixed point theorem, we investigate the existence of multiple positive periodic solutions for functional differential equations

$$\dot{x}(t) = A\big(t, x(t)\big)x(t) + \lambda f(t, x_t),$$

where $\lambda > 0$ is a parameter. Some easily verifiable sufficient criteria are established.


*Keywords:* Positive periodic solution; Functional differential equations; Krasnoselskii fixed point theorem

## 1. Introduction

Let $\mathbf{R} = (-\infty, +\infty)$, $\mathbf{R}_+ = [0, +\infty)$, $\mathbf{R}_- = (-\infty, 0]$ and $\mathbf{R}^\mathbf{n}_+ = \{(x_1, \dots, x_n)^T\colon x_i \geqslant 0,\ 1 \leqslant i \leqslant n\}$, respectively. For each $x = (x_1, x_2, \dots, x_n)^T \in \mathbf{R}^\mathbf{n}$, the norm of $x$ is defined as $|x| = \sum_{i=1}^n |x_i|$. Let $BC$ denote the Banach space of bounded continuous functions $\phi : \mathbf{R} \to \mathbf{R}^\mathbf{n}$ with the norm $\|\phi\| = \sup_{\theta \in R} \sum_{i=1}^n |\phi_i(\theta)|$, where $\phi = (\phi_1, \phi_2, \dots, \phi_n)^T$.

As it is well known, the existence of positive periodic solutions for functional equations was extensively studied, see [1,2,4–10] and references therein. One of effective approaches to fulfill

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such a problem is employing fixed point theorem, and some prior estimations of possible periodic solutions are obtained.

For example, Wang [9] studied the following functional equation:

$$x'(t) = a(t)g\big(x(t)\big)x(t) - \lambda b(t) f\big(x\big(t - \tau(t)\big)\big). \tag{1.1}$$

In [10], Ye, Fan and Wang discussed in detail a class of more general functional equations

$$\begin{aligned} &\dot{x}(t) = -a(t)x(t) + f\big(t, u(t)\big), \\ &\dot{x}(t) = a(t)x(t) - f\big(t, u(t)\big), \end{aligned} \tag{1.2}$$

where $u(t) = (x(g_1(t)), \ldots, x(g_{n-1})(t), \int_{-\infty}^{t} k(t-\theta)x(\theta)\,d\theta)$ and $\int_0^{+\infty} k(r)\,dr = 1$.

In paper [5], Jiang et al. investigated the existence, multiplicity and nonexistence of positive periodic solutions to a system of infinite delay equations

$$\dot{x}(t) = A(t)x(t) + \lambda f(t, x_t). \tag{1.3}$$

In the present paper, by utilizing the fixed point theorem due to Krasnoselskii, we aim to study the existence of multiple positive periodic solutions for the following functional differential equations

$$\dot{x}(t) = A\big(t, x(t)\big)x(t) + \lambda f(t, x_t) \tag{1.4}$$

in which $\lambda > 0$ is a parameter, $A(t, x(t)) = \mathrm{diag}[a_1(t, x(t)), a_2(t, x(t)), \ldots, a_n(t, x(t))]$, $a_i \in C(R \times R, R)$ is $\omega$-periodic, $f = (f_1, f_2, \ldots, f_n)^T$, $f(t, x_t)$ is a functional defined on $R \times BC$ and $f(t, x_t)$ is $\omega$-periodic whenever $x$ is $\omega$-periodic. If $x \in BC$, then $x_t \in BC$ for any $t \in R$, where $x_t$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in R$.

In our paper, we will discuss the existence of positive periodic solutions in more cases than the above mentioned papers and obtain some easily verifiable sufficient criteria.

Throughout the paper, we make the assumptions:

$(H_1)$ There exist continuous $\omega$-periodic functions $b_i(t), c_i(t)$, such that $b_i(t) \leqslant a_i(t, x(t)) \leqslant c_i(t)$, for $1 \leqslant i \leqslant n$.

$(H_2)$ $f_i(t, \phi_t) \int_0^{\omega} a_i(s, x(s))\,ds \leqslant 0$ for all $(t, \phi) \in R \times BC(R, R_+^n)$, $1 \leqslant i \leqslant n$.

$(H_3)$ $f(t, x_t)$ is a continuous function of $t$ for each $x \in BC(R, R_+^n)$.

$(H_4)$ For any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $[\phi, \psi \in BC, \|\phi\| \leqslant L, \|\psi\| \leqslant L, \|\phi - \varphi\| < \delta, 0 \leqslant s \leqslant \omega]$ imply $|f(s, \phi_s) - f(s, \psi_s)| < \varepsilon$.

In addition, the parameters in this paper are assumed to be not identically equal to zero.

To conclude this section, we summarize in the following a few concepts and results that will be needed in our arguments.

**Definition 1.1.** Let $X$ be Banach space and $E$ be a closed, nonempty subset of $X$, $E$ is said to be a cone if

(i) $\alpha u + \beta v \in E$ for all $u, v \in E$ and all $\alpha, \beta > 0$;
(ii) $u, -u \in E$ imply $u = 0$.

**Lemma 1.2** *(Krasnoselskii fixed point theorem).* [3] *Let $X$ be a Banach space, and let $E$ be a cone in $X$. Suppose $\Omega_1$ and $\Omega_2$ are open subsets of $X$ such that* $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$. *Suppose that*

$$T : E \cap (\bar{\Omega}_2 \setminus \Omega_1) \to E$$

*is a completely continuous operator and satisfies either*

(i) $\|Tx\| \geqslant \|x\|$ *for any* $x \in E \cap \partial\Omega_1$ *and* $\|Tx\| \leqslant \|x\|$ *for any* $x \in E \cap \partial\Omega_2$; *or*
(ii) $\|Tx\| \leqslant \|x\|$ *for any* $x \in E \cap \partial\Omega_1$ *and* $\|Tx\| \geqslant \|x\|$ *for any* $x \in E \cap \partial\Omega_2$.

*Then* $T$ *has a fixed point in* $E \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

## 2. Some lemmas

In this section, we make some preparations for the following sections.

Let

$$B_i = \max\{|b_i|, |c_i|\}, \quad 1 \leqslant i \leqslant n.$$

For $(t, s) \in R^2$, $1 \leqslant i \leqslant n$, we define

$$\sigma = \min\left\{\exp\left\{-2\int_0^{\omega} B_i(\tau)\,d\tau\right\}\frac{|\exp\{-\int_0^{\omega} c_i(\tau)\,d\tau\} - 1|}{|\exp\{-\int_0^{\omega} b_i(\tau)\,d\tau\} - 1|},\ 1 \leqslant i \leqslant n\right\} \tag{2.1}$$

and

$$G_i(t, s) = \frac{\exp\{-\int_t^s a_i(\tau, x(\tau))\,d\tau\}}{\exp\{-\int_0^{\omega} a_i(\tau, x(\tau)\,d\tau\} - 1}. \tag{2.2}$$

We also define

$$G(t, s) = \operatorname{diag}\big[G_1(t, s), G_2(t, s), \ldots, G_n(t, s)\big].$$

It is clear that $G(t, s) = G(t + \omega, s + \omega)$ for all $(t, s) \in R^2$ and by $(H_2)$,

$$G_i(t, s) f_i(u, \phi_u) \geqslant 0$$

for $(t, s) \in R^2$ and $(u, \phi) \in R \times BC(R, R_+^n)$.

A direct calculation shows that

$$m_i := \frac{\exp\{-\int_0^{\omega} B_i(\tau)\,d\tau\}}{|\exp\{-\int_0^{\omega} b_i(\tau)\,d\tau\} - 1|} \leqslant \big|G_i(t, s)\big| \leqslant \frac{\exp\{\int_0^{\omega} B_i(\tau)\,d\tau\}}{|\exp\{-\int_0^{\omega} c_i(\tau)\,d\tau\} - 1|} =: M_i. \tag{2.3}$$

Let

$$X = \big\{x \in C(R, R^n):\ x(t + \omega) = x(t),\ t \in R\big\}$$

and define

$$E = \big\{x \in X:\ x_i(t) \geqslant \sigma\|x_i\|,\ t \in [0, \omega],\ x = (x_1, x_2, \ldots, x_n)^T\big\}. \tag{2.4}$$

One may readily verify that $E$ is a cone.

We also assume

$(H_5)$ $\inf_{\|\phi\|=r} \int_0^{\omega} |f(s, \phi_s)|\,ds > 0$ for $\phi \in E$ and $r > 0$.

Moreover, define, for $r$ a positive number, $\Omega_r$ by

$$\Omega_r = \big\{x \in E:\ \|x\| < r\big\}.$$

Note that $\partial\Omega_r = \{x \in E:\ \|x\| = r\}$.

Let the map $T_\lambda : E \to E$ be defined by

$$(T_\lambda x)(t) = \lambda \int_t^{t+\omega} G(t,s) f(s, x_s)\, ds$$

for $x \in E$, $t \in R$, and let

$$(T_\lambda x) = \left(T_\lambda^1 x, T_\lambda^2 x, \dots, T_\lambda^n x\right)^T.$$

To prove the existence of positive solutions to Eq. (1.4), we first give the following lemmas.

**Lemma 2.1.** *Assume* $(H_1)$–$(H_4)$ *hold, then* $T_\lambda : E \to E$ *is well defined and* $T_\lambda$ *is compact and continuous.*

**Proof.** By $(H_3)$, for $x \in E$, we have $(T_\lambda x)$ is continuous in $t$ and

$$\begin{aligned}
(T_\lambda x)(t+\omega) &= \lambda \int_{t+\omega}^{t+2\omega} G(t+\omega, s) f(s, x_s)\, ds \\
&= \lambda \int_t^{t+\omega} G(t+\omega, v+\omega) f(v+\omega, x_{v+\omega})\, dv \\
&= \lambda \int_t^{t+\omega} G(t, v) f(v, x_v)\, dv = (T_\lambda x)(t).
\end{aligned}$$

Thus, $(T_\lambda x) \in X$. In view of (2.3), for $x \in E$, we have

$$\left\|T_\lambda^i x\right\| \leqslant \lambda M_i \int_0^\omega \left|f_i(s, x_s)\right| ds,$$

and

$$\left(T_\lambda^i x\right)(t) \geqslant \lambda m_i \int_0^\omega \left|f_i(s, x_s)\right| ds \geqslant \frac{m_i}{M_i} \left\|T_\lambda^i x\right\| \geqslant \sigma \left\|T_\lambda^i x\right\|.$$

Therefore, $(T_\lambda x) \in E$ and by $(H_4)$ it is easy to show that $T_\lambda$ is compact. □

**Lemma 2.2.** *Assume* $(H_1)$–$(H_4)$ *hold. Equation* (1.4) *is equivalent to the fixed point problem of* $T_\lambda$ *in* $E$.

**Proof.** If $x \in E$ and $T_\lambda x = x$, then

$$\begin{aligned}
\dot{x}(t) &= \frac{d}{dt}\left(\lambda \int_t^{t+\omega} G(t,s) f(s, x_s)\, ds\right) \\
&= \lambda G(t, t+\omega) f(t+\omega, x_{t+\omega}) - \lambda G(t,t) f(t, x_t) + A\big(t, x(t)\big) T_\lambda x(t) \\
&= \lambda\big[G(t, t+\omega) - G(t,t)\big] f(t, x_t) + A\big(t, x(t)\big) T_\lambda x(t) \\
&= A\big(t, x(t)\big) x(t) + \lambda f(t, x_t).
\end{aligned}$$

Thus $x$ is a positive $\omega$-periodic solution of (1.4). On the other hand, if $x$ is a positive $\omega$-periodic function, then $\lambda f(t,x_t)=\dot{x}(t)-A(t,x(t))x(t)$ and

$$\begin{aligned}(T_\lambda x)(t)&=\lambda\int_t^{t+\omega}G(t,s)f(s,x_s)\,ds\\&=\int_t^{t+\omega}G(t,s)\big[\dot{x}(s)-A\big(s,x(s)\big)x(s)\big]\,ds\\&=G(t,s)x(s)\big|_t^{t+\omega}+\int_t^{t+\omega}G(t,s)A\big(s,x(s)\big)x(s)\,ds-\int_t^{t+\omega}G(t,s)A\big(s,x(s)\big)x(s)\,ds\\&=x(t).\end{aligned}$$

Therefore, together with the proof of Lemma 2.1, we complete the proof. □

**Lemma 2.3.** *Assume* $(H_1)$–$(H_4)$ *hold and there exists* $\eta>0$ *such that*

$$\int_0^\omega\big|f(s,\phi_s)\big|\,ds\geqslant\eta\|\phi\|,\quad \text{for } \phi\in E.$$

*Then*

$$\|T_\lambda x\|\geqslant\lambda m\eta\|x\|,\quad \text{for } x\in E,$$

*where* $m=\min_{1\leqslant i\leqslant n}m_i$.

**Proof.** If $x\in E$, then

$$\big(T_\lambda^i x\big)(t)\geqslant\lambda m_i\int_t^{t+\omega}\big|f_i(s,x_s)\big|\,ds=\lambda m_i\int_0^\omega\big|f_i(s,x_s)\big|\,ds.$$

Thus, we have

$$\begin{aligned}\|T_\lambda x\|=\sup_{t\in R}\sum_{i=1}^n\big|\big(T_\lambda^i x\big)(t)\big|&\geqslant\sum_{i=1}^n\lambda m_i\int_0^\omega\big|f_i(s,x_s)\big|\,ds\\&\geqslant\lambda m\int_0^\omega\big|f(s,x_s)\big|\,ds\geqslant\lambda m\eta\|x\|.\end{aligned}$$ □

**Lemma 2.4.** *Assume* $(H_1)$–$(H_4)$ *hold and let* $r>0$. *If there exists a sufficiently small* $\varepsilon>0$ *such that*

$$\int_0^\omega\big|f(s,\phi_s)\big|\,ds\leqslant\varepsilon r,\quad \text{for } \phi\in E\cap\partial\Omega_r,$$

*then*

$$\|T_\lambda x\| \leqslant \lambda M\varepsilon\|x\|, \quad \textit{for } x \in E \cap \partial\Omega_r,$$

*where* $M = \max_{1\leqslant i\leqslant n} M_i$.

**Proof.** For any $x \in E \cap \partial\Omega_r$, we obtain

$$\begin{aligned}\|T_\lambda x\| &= \sup_{t\in R}\sum_{i=1}^{n}\left|\left(T_\lambda^i x\right)(t)\right| \\ &\leqslant \sup_{t\in R}\sum_{i=1}^{n}\lambda\int_t^{t+\omega}\left|G_i(t,s)\right|\left|f_i(s,x_s)\right|ds \\ &\leqslant \sum_{i=1}^{n}\lambda M_i\int_0^{\omega}\left|f_i(s,x_s)\right|ds \leqslant \lambda M\int_0^{\omega}\left|f(s,x_s)\right|ds \leqslant \lambda M\varepsilon r = \lambda M\varepsilon\|x\|.\end{aligned}$$

## 3. Main results

In order to state our main results, we assume the following limits exist and let

$$\sup f_0 = \lim_{\|\phi\|\downarrow 0}\sup_{\phi\in E}\frac{\int_0^\omega |f(s,\phi_s)|\,ds}{\|\phi\|}, \qquad \inf f_0 = \lim_{\|\phi\|\downarrow 0}\inf_{\phi\in E}\frac{\int_0^\omega |f(s,\phi_s)|\,ds}{\|\phi\|},$$

$$\sup f_\infty = \lim_{\|\phi\|\uparrow +\infty}\sup_{\phi\in E}\frac{\int_0^\omega |f(s,\phi_s)|\,ds}{\|\phi\|}, \qquad \inf f_\infty = \lim_{\|\phi\|\uparrow +\infty}\inf_{\phi\in E}\frac{\int_0^\omega |f(s,\phi_s)|\,ds}{\|\phi\|}.$$

Moreover, we list several assumptions

| | |
|---|---|
| $(P_1)$ $\inf f_0 = \infty$, | $(P_2)$ $\inf f_\infty = \infty$, |
| $(P_3)$ $\sup f_\infty = 0$, | $(P_4)$ $\sup f_0 = 0$, |
| $(P_5)$ $\sup f_0 = \alpha_1 \in \left[0, \frac{1}{\lambda M}\right)$, | $(P_6)$ $\inf f_\infty = \beta_1 \in \left(\frac{1}{\lambda m\sigma}, \infty\right)$, |
| $(P_7)$ $\inf f_0 = \alpha_2 \in \left(\frac{1}{\lambda m\sigma}, \infty\right)$, | $(P_8)$ $\sup f_\infty = \beta_2 \in \left[0, \frac{1}{\lambda M}\right)$, |

where $\sigma$ is defined in (2.1) and $m, M$ are defined in Lemmas 2.3 and 2.4.

**Theorem 3.1.** *If* $(P_1)$ *and* $(P_3)$ *hold, then* (1.4) *has at least one positive ω-periodic solution.*

**Proof.** By $(P_1)$, one can find $r_0 > 0$ such that

$$\int_o^\omega \left|f(s,\phi_s)\right|ds \geqslant \eta\|\phi\|, \quad \text{for } \phi \in E,\ 0 < \|\phi\| \leqslant r_0,$$

where the constant $\eta > 0$ satisfies $\lambda m\eta > 1$. Then by Lemma 2.3, we have

$$\|T_\lambda x\| \geqslant \lambda m\eta\|x\| > \|x\|, \quad \text{for } x \in E \cap \partial\Omega_{r_0}.$$

Again, by $(P_3)$, for any $0 < \varepsilon \leqslant 1/(2\lambda M)$, there exists $N_1 > r_0$ such that

$$\int_o^\omega \left|f(s, \phi_s)\right| ds \leqslant \varepsilon\|\phi\|, \quad \text{for } \phi \in E,\ \|\phi\| \geqslant N_1.$$

Choose

$$r_1 > N_1 + 1 + 2\lambda M \sup_{\substack{\|\phi\|<N_1 \\ \phi\in E}} \int_0^\omega \left|f(s, \phi_s)\right| ds.$$

If $x \in E \cap \partial\Omega_{r_1}$, then

$$\begin{aligned} \|T_\lambda x\| &\leqslant \lambda M \int_0^\omega \left|f(s, x_s)\right| ds \\ &= \lambda M\left(\int_{I_1} \left|f(s, x_s)\right| ds + \int_{I_2} \left|f(s, x_s)\right| ds\right) \\ &\leqslant \frac{r_1}{2} + \frac{\|x\|}{2} = \|x\|, \end{aligned}$$

where $I_1 = \{x \in E\colon\ \|x\| < N_1\}$, $I_2 = \{x \in E\colon\ \|x\| \geqslant N_1\}$. This implies that $\|T_\lambda x\| \leqslant \|x\|$ for any $x \in E \cap \partial\Omega_{r_1}$.

In conclusion, under the assumptions $(P_1)$ and $(P_3)$, $T_\lambda$ satisfies all the requirements in Lemma 1.2, then $T_\lambda$ has a fixed point in $E \cap (\bar{\Omega}_{r_1} \setminus \Omega_{r_0})$. By Lemma 2.2, we complete the proof. □

**Theorem 3.2.** *If* $(P_2)$ *and* $(P_4)$ *hold, then* (1.4) *has at least one positive* $\omega$*-periodic solution.*

**Proof.** By $(P_4)$, for any $0 < \varepsilon \leqslant 1/(\lambda M)$, there exists $r_2 > 0$ such that

$$\int_o^\omega \left|f(s, \phi_s)\right| ds \leqslant \varepsilon\|\phi\| \leqslant \varepsilon r_2, \quad \text{for } \phi \in E,\ 0 < \|\phi\| \leqslant r_2.$$

Then by Lemma 2.4, we have

$$\|T_\lambda x\| \leqslant \lambda M\varepsilon\|x\| \leqslant \|x\|, \quad \text{for } x \in E \cap \partial\Omega_{r_2}.$$

Next, by $(P_2)$, there exists $r_3 > r_2 > 0$ such that

$$\int_o^\omega \left|f(s, \phi_s)\right| ds \geqslant \eta\|\phi\|, \quad \text{for } \phi \in E,\ \|\phi\| \geqslant r_3,$$

where $\eta > 0$ is chosen so that $\lambda m\eta > 1$. It follows from Lemma 2.3 that

$$\|T_\lambda x\| \geqslant \lambda m\eta\|x\| > \|x\|, \quad \text{for } x \in E \cap \partial\Omega_{r_3}.$$

It follows from Lemma 1.2 that (1.4) has a positive $\omega$-periodic solution satisfying $r_2 \leqslant \|x\| \leqslant r_3$. □

In the following, we will introduce two extra assumptions to assure some basic theorems:

$(P_9)$ There exists $d_1 > 0$ such that $\int_o^\omega |f(s,\phi_s)|\,ds > d_1/(\lambda m)$, for $\sigma d_1 \leqslant \|\phi\| \leqslant d_1$.
$(P_{10})$ There exists $d_2 > 0$ such that $\int_o^\omega |f(s,\phi_s)|\,ds < d_2/(\lambda M)$, for $\|\phi\| \leqslant d_2$.

**Theorem 3.3.** *If $(P_3)$, $(P_4)$ and $(P_9)$ hold, then* (1.4) *has at least two positive $\omega$-periodic solutions $x^1$ and $x^2$ satisfying*

$$0 < \|x^1\| < d_1 < \|x^2\|.$$

**Proof.** By the assumption $(P_4)$, for any $0 < \varepsilon \leqslant 1/(\lambda M)$, there exists $r_4 < d_1$ such that

$$\int_o^\omega \big|f(s,\phi_s)\big|\,ds \leqslant \varepsilon\|\phi\|, \quad \text{for } \phi \in E,\ 0 < \|\phi\| \leqslant r_4,$$

then by Lemma 2.4, we obtain

$$\|T_\lambda x\| \leqslant \lambda M\varepsilon\|x\| \leqslant \|x\|, \quad \text{for } x \in E \cap \partial\Omega_{r_4}.$$

Likewise, from $(P_3)$, for any $0 < \varepsilon \leqslant 1/(2\lambda M)$, there exists $N_2 > d_1$ such that

$$\int_o^\omega \big|f(s,\phi_s)\big|\,ds \leqslant \varepsilon\|\phi\|, \quad \text{for } \|\phi\| \geqslant N_2.$$

Choose

$$r_5 > N_2 + 1 + 2\lambda M \max_{\substack{\|\phi\|<N_2\\ \phi\in E}} \int_0^\omega \big|f(s,\phi_s)\big|\,ds.$$

If $x \in E \cap \partial\Omega_{r_5}$, then

$$\begin{aligned}
\|T_\lambda x\| &\leqslant \lambda M \int_0^\omega \big|f(s,x_s)\big|\,ds \\
&= \lambda M\bigg(\int_{I_1} \big|f(s,x_s)\big|\,ds + \int_{I_2} \big|f(s,x_s)\big|\,ds\bigg) \\
&\leqslant \frac{r_5}{2} + \frac{\|x\|}{2} = \|x\|,
\end{aligned}$$

where $I_1 = \{x \in E\colon\ \|x\| < N_2\}$, $I_2 = \{x \in E\colon\ \|x\| \geqslant N_2\}$, which shows that $\|T_\lambda x\| \leqslant \|x\|$ for all $x \in E \cap \partial\Omega_{r_5}$.

Set $\Omega_{d_1} = \{x \in X\colon\ \|x\| < d_1\}$. Then, by $(P_9)$, for any $x \in E \cap \partial\Omega_{d_1}$, we have

$$\|T_\lambda x\| \geqslant \lambda m \int_0^\omega \big|f(s,x_s)\big|\,ds > \lambda m \frac{d_1}{\lambda m} = d_1 = \|x\|,$$

which yields $\|T_\lambda x\| > \|x\|$ for all $x \in E \cap \partial\Omega_{d_1}$. By Lemma 1.2, there exist two positive $\omega$-periodic solutions $x^1$ and $x^2$ satisfying $0 < \|x^1\| < d_1 < \|x^2\|$. This completes the proof. □

From the arguments in the above proof, we have the following consequence immediately.

**Corollary 3.4.** *If* $(P_1)$, $(P_2)$ *and* $(P_{10})$ *hold, then* (1.4) *has at least two ω-periodic solutions* $x^1$ *and* $x^2$ *satisfying*

$$0 < \|x^1\| < d_2 < \|x^2\|.$$

To obtain better results in this section, we give a more general criterion in the following, which plays an important role in the consequence.

**Theorem 3.5.** *Suppose that* $(P_9)$ *and* $(P_{10})$ *hold, then* (1.4) *has at least one positive ω-periodic solution* $x$ *with* $\|x\|$ *lying between* $d_2$ *and* $d_1$*, where* $d_1$ *and* $d_2$ *are defined in* $(P_9)$ *and* $(P_{10})$*, respectively.*

**Proof.** Without loss of generality, we may assume that $d_2 < d_1$. If $x \in E \cap \partial\Omega_{d_2}$, then by $(P_{10})$, one can get

$$\|T_\lambda x\| \leqslant \lambda M \int_0^\omega \bigl|f(s, x_s)\bigr|\, ds < \lambda M \frac{d_2}{\lambda M} = d_2 = \|x\|.$$

In particular, $\|T_\lambda x\| < \|x\|$ for all $x \in E \cap \partial\Omega_{d_2}$.

On the other hand, by $(P_9)$, one has

$$\|T_\lambda x\| \geqslant \lambda m \int_0^\omega \bigl|f(s, x_s)\bigr|\, ds > \lambda m \frac{d_1}{\lambda m} = d_1 = \|x\|,$$

which produces $\|T_\lambda x\| > \|x\|$ for all $x \in E \cap \partial\Omega_{d_1}$. Therefore, by Lemma 1.2, we can obtain the conclusion and this completes the proof. □

**Theorem 3.6.** *If* $(P_5)$ *and* $(P_6)$ *hold, then* (1.4) *has at least one positive ω-periodic solution.*

**Proof.** By assumption $(P_5)$, for any $\varepsilon = \frac{1}{\lambda M} - \alpha_1 > 0$, there exists a sufficiently small $d_2 > 0$ such that

$$\sup_{x \in E} \frac{\int_0^\omega |f(s, x_s)|\, ds}{\|x\|} < \alpha_1 + \varepsilon = \frac{1}{\lambda M}, \quad \text{for } \|x\| \leqslant d_2,$$

that is

$$\int_0^\omega \bigl|f(s, x_s)\bigr|\, ds < \frac{1}{\lambda M}\|x\| \leqslant \frac{d_2}{\lambda M}, \quad \text{for } \|x\| \leqslant d_2,$$

so, $(P_{10})$ is satisfied.

By assumption $(P_6)$, for $\varepsilon = \beta_1 - \frac{1}{\lambda m \sigma} > 0$, there exists a sufficiently large $d_1 > 0$ such that

$$\inf_{x \in E} \frac{\int_0^\omega |f(s, x_s)|\, ds}{\|x\|} > \beta_1 - \varepsilon = \frac{1}{\lambda m \sigma}, \quad \text{for } \|x\| \geqslant \sigma d_1,$$

that is,

$$\int_0^{\omega} \left|f(s,x_s)\right| ds > \frac{1}{\lambda m\sigma}\|x\| \geqslant \frac{1}{\lambda m\sigma}\sigma d_1 = \frac{d_1}{\lambda m}.$$

Therefore, $(P_9)$ holds. By Theorem 3.5 we complete the proof. □

**Theorem 3.7.** *If* $(P_7)$ *and* $(P_8)$ *hold, then* (1.4) *has at least one positive $\omega$-periodic solution.*

**Proof.** By assumption $(P_7)$, for any $\varepsilon = \alpha_2 - \frac{1}{\lambda m\sigma} > 0$, there exists a sufficiently small $d_1 > 0$ such that

$$\inf_{x\in E} \frac{\int_0^{\omega} |f(s,x_s)|\, ds}{\|x\|} > \alpha_2 - \varepsilon = \frac{1}{\lambda m\sigma}, \quad \text{for } 0 < \|x\| \leqslant d_1.$$

Therefore, we have

$$\int_0^{\omega} \left|f(s,x_s)\right| ds > \frac{1}{\lambda m\sigma}\sigma d_1 = \frac{d_1}{\lambda m}, \quad \text{for } \sigma d_1 \leqslant \|x\| \leqslant d_1.$$

That is, $(P_9)$ holds.

By consumption $(P_8)$, for $\varepsilon = \frac{1}{\lambda M} - \beta_2 > 0$, there exists a sufficiently large $d$ such that

$$\sup_{x\in E} \frac{\int_0^{\omega} |f(s,x_s)|\, ds}{\|x\|} < \beta_2 + \varepsilon = \frac{1}{\lambda M}, \quad \text{for } \|x\| > d.$$

In the following, we consider two cases to prove $(P_{10})$ to be satisfied: $\sup_{x\in E}\int_0^{\omega} |f(s,x_s)|\, ds$ bounded and unbounded. The bounded case is clear. If $\sup_{x\in E}\int_0^{\omega} |f(s,x_s)|\, ds$ is unbounded, then there exists $y \in \mathbf{R}_+^{\mathbf{n}}$, $\|y\| = d_2 > d$, such that

$$\int_0^{\omega} \left|f(s,x_s)\right| ds \leqslant \int_0^{\omega} \left|f(s,y_s)\right| ds, \quad \text{for } 0 < \|x\| \leqslant \|y\| = d_2.$$

Since $\|y\| = d_2 > d$, then we have

$$\int_0^{\omega} \left|f(s,x_s)\right| ds \leqslant \int_0^{\omega} \left|f(s,y_s)\right| ds < \frac{1}{\lambda M}\|y\| = \frac{d_2}{\lambda M}, \quad \text{for } 0 < \|x\| \leqslant d_2,$$

which implies the condition $(P_{10})$ holds. Therefore, by Theorem 3.5 we complete the proof. □

**Theorem 3.8.** *Suppose that* $(P_6)$, $(P_7)$ *and* $(P_{10})$ *hold, then* (1.4) *has at least two positive $\omega$-periodic solutions $x^1$ and $x^2$ satisfying* $0 < \|x^1\| < d_2 < \|x^2\|$, *where $d_2$ is defined in* $(P_{10})$.

**Proof.** From $(P_6)$ and the proof of Theorem 3.6, we know that there exists a sufficiently large $d_1 > d_2$, such that

$$\int_0^{\omega} \left|f(s,x_s)\right| ds > \frac{d_1}{\lambda m}, \quad \text{for } \sigma d_1 \leqslant \|x\| \leqslant d_1.$$

From $(P_7)$ and the proof of Theorem 3.7, we can find a sufficiently small $d_1^* \in (0, d_2)$ such that

$$\int_0^\omega \left|f(s, x_s)\right| ds > \frac{d_1^*}{\lambda m}, \quad \text{for } \sigma d_1^* \leqslant \|x\| \leqslant d_1^*.$$

Therefore, from the proof of Theorem 3.5, there exist two positive solutions $x^1$ and $x^2$ satisfying $d_1^* < \|x^1\| < d_2 < \|x^2\| < d_1$. □

From the arguments in the above proof, we have the following consequence, too.

**Corollary 3.9.** *Suppose that* $(P_5)$, $(P_8)$ *and* $(P_9)$ *hold, then* (1.4) *has at least two positive* $\omega$*-periodic solutions* $x^1$ *and* $x^2$ *satisfying* $0 < \|x^1\| < d_2 < \|x^2\|$, *where* $d_1$ *is defined in* $(P_9)$.

**Theorem 3.10.** *If* $(P_1)$ *and* $(P_8)$ *hold, then* (1.4) *has at least one positive* $\omega$*-periodic solution.*

**Proof.** From the assumption $(P_1)$ and the proof of Theorem 3.1, we know that $\|T_\lambda x\| \geqslant \|x\|$ for all $x \in E \cap \partial\Omega_{r_0}$.

From $(P_8)$ and Theorem 3.7, as $\|x\| \leqslant r_1$, we know that $\int_0^\omega |f(s, x_s)|\, ds < \frac{r_1}{\lambda M}$ and

$$\|T_\lambda x\| \leqslant \lambda M \int_0^\omega \left|f(s, x_s)\right| ds < \lambda M \frac{r_1}{\lambda M} = r_1 = \|x\|,$$

which implies $\|T_\lambda x\| < \|x\|$ for all $x \in E \cap \partial\Omega_{r_1}$. This completes the proof. □

Similar to Theorem 3.10, one immediately has the following consequences.

**Theorem 3.11.** *If* $(P_2)$ *and* $(P_5)$ *hold, then* (1.4) *has at least one positive* $\omega$*-periodic solution.*

**Theorem 3.12.** *If* $(P_3)$ *and* $(P_7)$ *hold, then* (1.4) *has at least one positive* $\omega$*-periodic solution.*

**Theorem 3.13.** *If* $(P_4)$ *and* $(P_6)$ *hold, then* (1.4) *has at least one positive* $\omega$*-periodic solution.*

**Theorem 3.14.** *If* $(P_1)$, $(P_6)$ *and* $(P_{10})$ *hold, then* (1.4) *has at least two positive* $\omega$*-periodic solutions* $x^1$ *and* $x^2$ *satisfying*

$$0 < \left\|x^1\right\| < d_2 < \left\|x^2\right\|,$$

*where* $d_2$ *is defined in* $(P_{10})$.

**Proof.** Let $\Omega_{r_*} = \{x \in X\colon \|x\| < r_*\}$, where $r_* < d_2$. By assumption $(P_1)$ and the proof of Theorem 3.1, we know $\|T_\lambda x\| \geqslant \|x\|$ for all $x \in E \cap \partial\Omega_{r_*}$.

Let $\Omega_{d_1} = \{x \in X\colon \|x\| < d_1\}$. By the assumption $(P_6)$ and the proof of Theorem 3.6, we can see that $\int_0^\omega |f(s, x_s)|\, ds > \frac{d_1}{\lambda m}$ for $\sigma d_1 \leqslant \|x\| \leqslant d_1$. Incorporating $(P_{10})$ and the proof of Theorem 3.5, we know that there exist two positive $\omega$-periodic solutions $x^1$ and $x^2$ satisfying $0 < \|x^1\| < d_2 < \|x^2\|$. □

**Theorem 3.15.** *If* $(P_2)$, $(P_7)$ *and* $(P_{10})$ *hold, then* (1.4) *has at least two positive* $\omega$*-periodic solutions* $x^1$ *and* $x^2$ *satisfying*

$$0 < \|x^1\| < d_2 < \|x^2\|,$$

*where* $d_2$ *is defined in* $(P_{10})$.

**Theorem 3.16.** *If* $(P_3)$, $(P_5)$ *and* $(P_9)$ *hold, then* (1.4) *has at least two positive* $\omega$*-periodic solutions* $x^1$ *and* $x^2$ *satisfying*

$$0 < \|x^1\| < d_1 < \|x^2\|,$$

*where* $d_1$ *is defined in* $(P_9)$.

**Theorem 3.17.** *If* $(P_4)$, $(P_8)$ *and* $(P_9)$ *hold, then* (1.4) *has at least two positive* $\omega$*-periodic solutions* $x^1$ *and* $x^2$ *satisfying*

$$0 < \|x^1\| < d_1 < \|x^2\|,$$

*where* $d_1$ *is defined in* $(P_9)$.

## References

[1] S.-N. Chow, Existence of periodic solutions of autonomous functional differential equations, J. Differential Equations 15 (1974) 350–378.
[2] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, New York, 1985.
[3] M.A. Krasnoselskii, Positive Solution of Operator Equation, Noordhoff, Gröningen, 1964.
[4] D.Q. Jiang, J.J. Wei, B. Zhang, Positive periodic solutions of functional differential equations and population models, Electron. J. Differential Equations 71 (2002) 1–13.
[5] D.Q. Jiang, D. O'Regan, R.P. Agarwal, X.J. Xu, On the number of positive periodic solutions of functional differential equations and population models, Math. Models Methods Appl. Sci. 15 (4) (2005) 555–573.
[6] G. Makay, Periodic solutions of dissipative functional differential equations, J. Tohoku Math. 46 (1994) 417–426.
[7] M.J. Ma, J.S. Yu, Existence of multiple positive periodic solutions for nonlinear functional difference equations, J. Math. Anal. Appl. 305 (2005) 483–490.
[8] S.G. Peng, S.M. Zhu, Positive periodic solutions for functional differential equations with infinite delay, Chinese Ann. Math. Ser. A 25 (3) (2004) 285–292 (in Chinese).
[9] H.Y. Wang, Positive periodic solutions of functional differential equations, J. Differential Equations 202 (2004) 354–366.
[10] D. Ye, M. Fan, H.Y. Wang, Periodic solutions for scalar functional differential equations, Nonlinear Anal. 62 (2005) 1157–1181.